WEIGHTING FUNCTION IN THE
BEHAVIORAL PORTFOLIO THEORY.

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Abstract

The Behavioral Portfolio Theory (BTP) developed by Shefrin and Statman (2000) considers a probability weighting function rather than the real probability distribution used in Markowitz’s Portfolio Theory (1952). The optimal portfolio of a BTP investor, which consists in a combination of bonds and lottery ticket, can differ from the perfectly diversified portfolio of Markowitz. We found that this particular form of portfolio is not due to the weighting function. In this article we explore the implication of weighting function in the portfolio construction. We prove that the expected wealth criteria (used by Shefrin and Statman), even if the objective probabilities were deformed, is not a sufficient condition for obtaining significantly different forms of portfolio. Not only probabilities but also future outcomes have to be transformed.

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1 Introduction

The expected wealth or expected return criteria is an intuitive criteria used for decision making under uncertainty. In fact, economic analysis of choice under risky alternatives has been dominated by the expected utility (EU) model proposed by Von Neumann and Morgenstern in 1947. Following this framework the preferences of an individual are determined by an increased utility function \( U \). We assume a concave utility function in order to take into account the risk aversion of individuals. In the particular case where \( U \) is a linear function, an individual using the expected utility criteria can be considered as an expected wealth maximiser. In the same way, the Markowitz (1952) portfolio theory, (the most used on portfolio analysis), is based on the mean-variance approach. More precisely Markowitz suggests that when investors choose their portfolio they maximize the expected return subject to standard deviation which is maintained on a constant level. In this approach the standard deviation is used to measure the degree of portfolio risk.

Other models (e.g. Lopes, 1987; Tversky and Kahneman, 1992; Shefrin and Statman, 2000) are also founded on the expected wealth criteria, but, opposite to classical approach, they do not require that individual preferences are linear in probability. Drawing on experimental evidences (Edwards, 1953, 1954; Allais, 1953; Kahneman, Slovic and Tversky, 1982), the apparent over-weighting of low probability events with extreme consequences has been put forward. This observation was the starting point of the first wave of the “non-linear models”. Their main idea was to replace the objective probabilities by decision weights. The decision weight associated with a given event was related to the objective probability by a concave or convex or S-shaped function \( w \). Unfortunately, within the probability - weight approach, the violation of the concept of dominance arises except in the special case where the model is equivalent to EU. A solution to this problem was offered by Quiggin (1982) who applies the weighting function, not to the probabilities of individual events themselves, but to the cumulative distribution function. More precisely, if the future portfolio’s payments are given by \( 0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \) with corresponding objectives probabilities \( p_1, \ldots, p_n \), the weight associated to each payment is given by:
\[ q_i = w(p_i), \]
\[ q_i = w(p_1 + p_2 + \ldots + p_i) - w(p_1 + p_2 + \ldots p_{i-1}) = w(F(x_i)) - w(F(x_{i-1})) \text{ for } i \geq 2 \]

where \( F(x_i) = P(W \leq x_i) \) is the cumulative distribution function of the random variable \( W \) denoting a future wealth. The weighting function \( w : [0,1] \rightarrow [0,1] \) has to satisfy \( w(0) = 0 \) and \( w(1) = 1 \).

The non linear models (Quiggin, 1982; Yaari, 1987; Tversky and Kahneman, 1992), offer an alternative notion of risk aversion. In fact, within the expected utility theory, the only explanation for risk aversion is that the utility function is concave. The non linear models combine two different concepts: they distinguish between the attitude towards outcomes and the attitude towards probabilities. The manner to transform the objectives probabilities characterizes pessimists or optimists. A pessimistic person is someone who for worst outcomes adopts decision weights higher than the real probability and who for better outcomes adopts decision weights smaller than the real probability. To put the matter somewhat differently, the transformed probabilities of a pessimist yields an expected value lower than the mathematical expectation (Yaari M., 1987).

The probability weighting allows to explain, for instance, the popularity of lottery tickets over the centuries\(^1\). This phenomenon can not be explained by EU that supposes a risk aversion attitude. In fact, the jackpot of a bet associated with the smallest chance to be won seems to be attractive for an optimistic person who overweighs the real probability of gain. Moreover, the possibility to combine the optimistic attitude with risk aversion offers the explanation of so called Friedman and Savage (1948) paradox. The authors point out the fact that most people who purchase insurance policies buy also lottery tickets. The coexistence of gambling and insurance has been observed also in portfolio management (Blume M, and Friend I, 1975; Jorion P, 1994). Late on, in 2000 Shefrin and Statman developed the Behavioral Portfolio Theory (BPT ) who seems to be able to take into

\(^1\) Leonnet (1936); Pfiffelmann, Roger (2005).
account this “double” behavior. The BPT- agent maximizes expected wealth for transformed probabilities:

\[
\text{Max } E_h(W)
\]

subject to the constraint that

\[
P(W < A) \leq \alpha
\]

where \( A \) is the aspiration level which a person seeks to reach. Nevertheless, he can bear the failure with a small probability that must not exceed \( \alpha \). This means that BPT investor maximizes expected wealth on a particular set of portfolio that satisfied the security constraint. As a result, the optimal portfolio of BPT investor is a combination of bonds or a riskless asset and a lottery ticket. In the fist stage, the investor seeks to be in security and he buys the bonds or a riskless asset in order to reach his aspiration level \( A \) with the maximum chance of failure \( \alpha \). In the second stage the investor is able to take risk with the residual wealth. In consequence, the BTP optimal portfolio can differ from the perfectly diversified portfolio of Markowitz\(^2\) who suggests that individual seeks to reduce risk as much as possible.

In the BPT model, risk aversion is taken into account by the constraint \( P(W < A) \leq \alpha \) that determines the set of security portfolios. It is important to underline that BTP investor does not transform the future outcomes comparing with other non linear models (for instance Quiggin, 1982; Tversky and Kahneman, 1992). In terms of EU theory this means that his utility function is linear. In other words, the BPT - investor maximizes his expected wealth on a particular set of portfolios for transformed probabilities. In fact, the weighting function is a one of the central point in the Shefrin and Statman theory. Nevertheless, we will prove hereafter that the particular form of the BPT optimal portfolio is not the consequence of probability transformation. More precisely we will compare two investors: the first one, called VNM – investor (to pay a tribute to Von Neumann and Morgenstern), maximizes the expected wealth without probabilities transformation. The second one, called BPT – investor, transforms the objective probabilities by a weighting function before maximizing his expected wealth. Given that \( A = 0 \) for the VNM – investor as well as for

\(^2\) The recent studies (Levy, 2004; Broihanne, Merli and Roger, 2006) prove that BPT and mean –variance efficient frontiers coincide in the case of normal distribution of return.
the BPT investor, the security constraint is always satisfied. This hypothesis permits to establish the real impact of probability transformation on the portfolio selection because the impact due to security constraint was eliminated (the inequality $P(W < 0) \leq \alpha$ is always verified). Finally, we will compare the optimal portfolio of a VNM – investor who maximizes $E(W)$ with the optimal portfolio of a BTP investor maximizing $E_q(W)$.

The paper is organized as follows: in section 2 we consider a very simple market with two contingent claims and two assets. First we remind the well know result of a VNM - agent maximization, in particular that his optimal portfolio is a risky portfolio. Afterwards, we will see that at optimum the BPT investor can choose a riskless asset. In section 3 we test this outcome in the case of three assets. We establish, using an analytical and graphical approach, a necessary and sufficient condition in order that the BPT agent invests in a riskless asset. We finally prove that this condition can never be satisfied on the set of all accessible portfolios. We give some concluding observations in section 4.

2 Case of two assets $n = 2$

Let us remind the results of expected wealth maximization for a VNM – investor.

2.1 VNM – agent’s choice

Consider a market with two contingent claims $e_1$ and $e_2$ in date zero. At date one a state $i$ contingent claim pays one unit of consumption if state $i$ occurs and zero otherwise. Let the probability of the state $i$ be $p_i$, $i = 1, 2$. Let $n_i$ note supply of asset $e_i$ and $\pi$ note his price in date 0.

Suppose the investor has a portfolio $W_0 = (x_1, x_2)$ at date 0. The VNM- agent seeks to maximize date one expected wealth $W = (x_1, x_2)$, subject to budget constrain. His program is given by:

$$\text{Max } (x_1 p_1 + x_2 p_2)$$

---

3 In order to except trivial case we suppose $p_1 \neq p_2$. Further, in the case with more than two assets, we will suppose that the law of probability is not uniform.
This program characterizes a risk neutral person, his indifference curves are parallel straight lines with slope equal \(-\frac{P_1}{P_2}\). The slope of the budget line is \(-\frac{\pi_1}{\pi_2}\). Thus, there are three possibilities: both slopes are equal or one of them is more or less steeper than another one. Figure 1.1 shows the first possibility when both slopes are equal \(\frac{\pi_1}{\pi_2} = \frac{P_1}{P_2}\). In this case the budget line coincides with an indifference curve and the agent is indifferent between all portfolios belonging to the budget line. In other words, a riskless portfolio \((x_1 = x_2)\) and a risky portfolio, for example \((x_1, 0)\) or \((0, x_2)\), generate the same level of satisfaction.

Figures 1.2 and 1.3 show respectively two other cases \(\frac{\pi_1}{\pi_2} < \frac{P_1}{P_2}\) and \(\frac{\pi_1}{\pi_2} > \frac{P_1}{P_2}\), in both the agent prefers to invest in only one of the two assets. The point \(W^*\) illustrates the optimal portfolio, it equals \((x_1, 0)\) if \(\frac{\pi_1}{\pi_2} < \frac{P_1}{P_2}\) (or, \(\frac{\pi_1}{P_1} < \frac{\pi_2}{P_2}\)) and \((0, x_2)\) if \(\frac{\pi_1}{\pi_2} > \frac{P_1}{P_2}\) (or \(\frac{\pi_1}{P_1} > \frac{\pi_2}{P_2}\)). It means that it is optimal to purchase a portfolio with the lowest price per unit of probability.
\( \mathcal{P}^+ \) is the set of all portfolios available for purchase\(^4\). We note that the optimal portfolio of a VNM – investor belongs to the boundary of the set \( \mathcal{P}^+ \), hence it is a risky portfolio. In the particular case 1.1, there are two risky portfolios belonging to the boundary that can be chosen by the agent.

2.2 BTP- investor’s choice

Consider a BPT – agent who maximizes the expected wealth as a VNM – investor but who deforms the real probability according to the Quiggin’s (1984,1987) rule. We study how the results obtained earlier will be affected by this psychological aspect. The new program is given by:

\[
\begin{align*}
\text{Max } & x_1q_1 + x_2q_2 \\
\text{s.c } & (x_1 - x_{01})\pi_1 + (x_2 - x_{02})\pi_2 = 0 \\
& x_1 \geq 0, \ x_2 \geq 0
\end{align*}
\]

where \( q_1 \) and \( q_2 \) are the weights associated to the real probabilities \( p_1 \) et \( p_2 \). Let’s denote \( S_1 = \{(x_1,x_2)/x_1 \leq x_2\} \) and \( S_2 = \{(x_1,x_2)/x_2 \leq x_1\} \). The weights are not determined \textit{a priori} in the same manner in two sets. In fact,

\[
q_1 = w(p_1) \quad \text{and} \quad q_2 = w(p_1 + p_2) - w(p_1) = 1 - w(p_1) \quad \text{if } x_1 < x_2
\]

and

\[
q_2 = w(p_2) \quad \text{and} \quad q_1 = w(p_2 + p_1) - w(p_2) = 1 - w(p_2) \quad \text{if } x_2 < x_1
\]

\( w \) is a weighting function such as \( w(0) = 0 \) and \( w(1) = 1 \) which shows the level of optimism or pessimism of the individual. Thus, the set of all portfolios \( \mathcal{P}^+ \) is decomposed in two parts \( S_1 \) and \( S_2 \). The slope of the indifference curves \( -\frac{q_1}{q_2} \) equals \( -\frac{w(p_1)}{1-w(p_1)} \) in set the \( S_1 \)

\(^4\) \( \mathcal{P}^+ = \{(x_1,x_2)/x_1 \geq 0, \ x_2 \geq 0\} \)

\(^5\) In the trivial case \( x_1 = x_2 \), the only one optimal solution is to invest the amount \( \frac{W_0}{\pi_1 + \pi_2} \) in each asset.
and $\frac{1 - w(p_2)}{w(p_2)}$ in set the $S_2$. Nevertheless, we can prove, because of $w(p_1) + w(p_2) = 1$, that two slopes are equal:

$$\frac{w(p_1)}{1 - w(p_1)} = \frac{1 - w(p_2)}{w(p_2)}.$$  

In fact,

$$w(p_1)w(p_2) = (1 - w(p_1))(1 - w(p_1))$$

$$w(p_1)w(p_2) = 1 - w(p_1) - w(p_2) + w(p_1)w(p_2)$$

$$w(p_1) + w(p_2) = 1.$$  

The latter equality must be satisfied for any weighting function $w$. It means that the indifference curves of a person who deforms the real probabilities are the straight parallel lines with slope $-\frac{q_1}{q_2} = \frac{w(p_1)}{1 - w(p_1)} = \frac{1 - w(p_2)}{w(p_2)}$. Thus, the optimal portfolio of a BPT – agent has the same form that the portfolio chosen by a VNM – agent; more exactly both belong to the boundary of the set $\wp_{S_2}$. This does not mean that they will choose the same portfolio because, à priori, the indifference curve’s slopes differ: $-\frac{q_1}{q_2} \neq -\frac{p_1}{p_2}$. However, there is no significant difference between the behavior of the two investors in a way that a BTP agent also invests in a risky asset.

This result has been drawn thinks to the equality $w(p_1) + w(p_2) = 1$ which is true for all $n$. But, if $n > 2$ its meaning is not the same. More precisely, this does not allow to conclude that the indifference plans associated to different sets resulted from change in probability are parallel.

In fact, let us consider now a market with three contingent claims, $n = 3$. Let’s note $S_{ijk} = \{W \mid 0 \leq x_i < x_j < x_k; \ i, j, l = 1, 2, 3 \ et \ i \neq j \neq k\}$. So, the set $\wp_{S_3}$ of all portfolios is divided into six areas $S_{ijk}$. The weights $q_i$, $q_j$ and $q_k$ corresponding to each area differ and they are determinated according to Quiggin’s rule:

$$q_i = w(p_i),$$

$$q_j = w(p_i + p_j) - w(p_i),$$

$$q_k = w(p_i + p_j + p_k) - w(p_i + p_j) = 1 - w(p_i + p_j).$$  

8
The indifference plan equation is:

\[ x_i q_i + x_j q_j + x_k q_k = \text{const} \]

We can prove that the plans associated to different \( S_{jk} \) are not parallel. For instance, let us consider \( S_{123} \) and \( S_{132} \). The equations of indifference plans are:

\[
\begin{align*}
    x_i w(p_i) + x_2 (w(p_1 + p_2) - w(p_i)) + x_3 (1 - w(p_1 + p_2)) &= \text{const} \quad \text{in } S_{123} \\
    x_i w(p_i) + x_2 (w(p_1 + p_2) - w(p_i)) + x_3 (1 - w(p_1 + p_3)) &= \text{const} \quad \text{in } S_{132}
\end{align*}
\]

The plans will be parallel if and only if

\[
\frac{w(p_i)}{w(p_i)} = \frac{w(p_1 + p_2) - w(p_i)}{w(p_1 + p_3) - w(p_i)} = \frac{1 - w(p_1 + p_2)}{1 - w(p_1 + p_3)}.
\]

Nevertheless, none one of these two inequalities can be verified because \( w(p_1 + p_2) \neq w(p_1 + p_3) \).

We will see that this fact can affect greatly the portfolio choice. In order to explore the case with three contingent claims we must use a graphical approach that is difficult to represent. Thus, let us consider a market with two assets and suppose \( w(p_1) + w(p_2) \neq 1 \). This case is not really realistic but it can help us to understand what happens in the market with three and more assets because this inequality means no parallel indifference curves.

2.3 Graphical approach

If \( w(p_1) + w(p_2) \neq 1 \), there are two possibilities: indifference curves in the set \( S_1 \) are more or less steep than indifference curves in the set \( S_2 \). We consider only the first case. The second one is not very interesting because it leads to an optimal portfolio similar to the one chosen by a VNM – agent\(^6\).

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\(^6\) See the annex 2.
When we maximize expected wealth, we are in a particular case of expected utility theory: utility function is linear and characterizes a risk neutral person. At the same time, the figure 2 describes an individual who is risk averse. It is important to underline that the form of the indifference curve is the result of the weighting function and not of the utility function.

Figure 2 shows different position of the budget line. In both cases 1.1 and case 1.2 the optimal solution is similar to one that has been drawn for an VMN – investor, i.e portfolio belonging to the boundary of set the $\square_+^{1.2}$. The investor follows the same strategy: he invests in the portfolio which has the lowest price per unit of probability\(^7\).

\[^7\text{See the annex 1 for proof.}\]
The case 1.3 is more interesting because the riskless asset is the optimal solution. It means that the risk neutral agent who deforms the real probabilities in the manner represented on figure 3 case 1.3 chooses a riskless asset at optimum rather than a risky asset that would have been selected by a VMN-agent. More exactly, a VMN – agent can also invest in a riskless asset but only in the case when he is indifferent between all portfolios from the budget line. While for a BPT investor the riskless asset is the only one portfolio that can be chosen at optimum. We would like to emphasize this fact as an essential difference in the behavior of two investors.
3.2 Case of three assets

Let us consider a market with three contingent claims and an investor who deforms the real probabilities, his program is as follows

\[
\text{Max } x_1q_1 + x_2q_2 + x_3q_3 \\
\text{s.c } (x_1 - x_{o1})\pi_1 + (x_2 - x_{o2})\pi_2 + (x_3 - x_{o3})\pi_3 = 0
\]

\[x_i \geq 0; \quad x_2 \geq 0; \quad x_3 \geq 0.\]

where \(q_1\), \(q_2\) et \(q_3\) denote weights that are determined according to a Quiggin’s rule. As we said before the set of all portfolios is divided into 6 parts \(S_{ijk}\). Hence, the program 6 must be resolved in each \(\overline{S}_{ijk}\) :

\[
\text{Max } x_1q_i + x_jq_j + x_kq_k \\
\text{s.c } (x_i - x_{o1})\pi_i + (x_j - x_{o2})\pi_j + (x_k - x_{o3})\pi_k = 0
\]

\[x_i \geq 0; \quad x_i \leq x_j; \quad x_j \leq x_k.\]

The corresponding Lagrangian \(L_{ijk}\) is given by :

\[
L_{ijk} = x_iq_i + x_jq_j + x_kq_k - \lambda((x_i - x_{o1})\pi_i + (x_j - x_{o2})\pi_j + (x_k - x_{o3})\pi_k)
\]

\[+ \mu_i(x_i - x_j) - \mu_3(x_j - x_k)\]

where \(\mu_i \geq 0, \mu_2 \geq 0\) and \(\mu_3 \geq 0\). We seek to resolve the following system :

\[
\begin{align*}
q_i - \lambda\pi_i + \mu_i - \mu_2 &= 0 \\
q_j - \lambda\pi_j + \mu_2 - \mu_3 &= 0 \\
q_k - \lambda\pi_k + \mu_3 &= 0 \\
\text{c.b} \\
\mu_i x_i &= 0; \quad \mu_2(x_i - x_j) = 0; \quad \mu_3(x_j - x_k) = 0; \\
x_i \geq 0; \quad x_i \leq x_j; \quad x_j \leq x_k.
\end{align*}
\]

In the first place note that if both constraints \(x_i \leq x_j\) and \(x_j \leq x_k\) are not saturated we obtain a solution similar to that would be obtained in the case of a VNM - investor. At the same time, if at least one of these constraints is saturated, for example \(\mu_2 > 0\), the system 7 does not comply with a program 6. In fact, \(\mu_2 > 0\) means to seek a solution so that \(x_i = x_j\). But, in the plan \(x_i = x_j\) the weights are not determined in the same manner that in the set

\[\overline{S}_{ijk} = \{W/0 \leq x_i \leq x_j \leq x_k; \quad i, j, l = 1, 2, 3 \text{ et } i \neq j \neq k\}\]

is the complement of \(S_{ijk}\) in \(\square^3\) .
$S_{yk}$ where $x_i$ is strictly smaller than $x_j$. In fact, if $\mu_2 > 0$ the three - dimensions program 6 must be replaced by one of two - dimensions program:

$$\begin{align*}
\text{Max} & \quad x_y q_y + x_i q_k \\
\text{s.c} & \quad (x_y - x_{0y})\pi_i + (x_j - x_{0j})\pi_j + (x_k - x_{0k})\pi_k = 0 \\
\quad & \quad x_y \geq 0; x_y \leq x_k
\end{align*}$$

where $x_{ij}$ represents the quantity of assets $e_1$ and $e_2$ at optimum and the weights are determined by

$$q_y = w(p_i + p_j) \quad \text{and} \quad q_k = 1 - q_y.$$ 

If both constrains $x_i \leq x_j$ and $x_j \leq x_k$ are simultaneously saturated we have $\mu_2 > 0$ and $\mu_3 > 0$ in the system 7. Thus, we seek a solution that must satisfy $x_i = x_j = x_k$ corresponding to a riskless portfolio, the most interesting case. On the other hand, the assumption that two constraints are saturated leads to one - dimension program whose solution is well known: investment of the same amount $\frac{W_0}{\pi_1 + \pi_2 + \pi_3}$ in each asset. In other words, the perception in one - dimension cannot allow to establish any conditions about decision weights from $S_{yk}$ (corresponding to three - dimension) which lead an agent to invest in a riskless asset. Nevertheless, our goal is to study the case of a riskless asset in $\Box^+_3$ and determine these conditions. Thus, because we cannot reach our goal with the analytical approach, we will use the graphical approach. We will establish a necessary and sufficient condition so that a riskless asset would be the optimal solution for a BTP investor.

### 3.3 Graphical Approach. Necessary and Sufficient Condition.

The set $S_{yk}$ is a subset of $\Box^+_3$ enclosed between two plans $x_i = x_j$ and $x_j = x_k$. Their intersection is the straight line $x_i = x_j = x_k$. The form of the optimal portfolio depends on the position of the budget plan regarding to the indifference plans. The riskless asset is the optimal solution if, in both sets $x_i = x_j$ and $x_j = x_k$, the indifference curves are steeper than
the budget line. In this case, and only in this case, the intersection of the budget plan with the highest indifference plan in $S_{ijk}$ is a point from straight line $x_i = x_j = x_k$. The equation of indifference plan is given by

$$x_i q_i + x_j q_j + x_k q_k = \text{const}$$

and the equation of the budget constraint is

$$x_i \pi_i + x_j \pi_j + x_k \pi_k - W_0 = 0$$

If $x_i = x_j$ we have

$$x_i (q_i + q_j) + x_k q_k = \text{const}$$

$$x_i (\pi_i + \pi_j) + x_k \pi_k - W_0 = 0$$

The fact that the slope of indifference curve is bigger than the one of the budget line is shown as follows:

$$\frac{q_i + q_j}{q_k} > \frac{\pi_i + \pi_j}{\pi_k}$$  \hspace{1cm} (8)

If $x_j = x_k$ we have

$$x_i q_i + x_j (q_j + q_k) = \text{const}$$

$$x_i \pi_i + x_j (\pi_j + \pi_k) - W_0 = 0$$

and our condition takes the following form:

$$\frac{q_i}{q_j + q_k} > \frac{\pi_i}{\pi_j + \pi_k}$$  \hspace{1cm} (9)

Note that the riskless asset is also the optimal portfolio in the case of equality in the equations 8 and 9. This particular case is not very interesting because the agent is indifferent between a riskless asset and any other one from the set $x_i = x_j$ (equation 8) or from the set $x_j = x_k$ (equation 9). But we are interested in a situation where the riskless asset is only one possible solution. We consider, thus the strict inequalities in 8 and 9.

We can prove that if 8 and 9 are verified then a riskless asset is the optimal solution of system 7. Hence, 9 and 8 together are a sufficient condition. In fact, let us suppose that equation 8 is satisfied. According to first three equations of 7 we have:

$$\frac{q_i + q_j}{q_k} = \frac{\lambda (\pi_i + \pi_j) - \mu_i + \mu_j}{\lambda \pi_k - \mu_j} > \frac{\pi_i + \pi_j}{\pi_k}.$$
Or

\[ \lambda(\pi_i + \pi_j)\pi_k - \mu_i\pi_k + \mu_j\pi_k > \lambda(\pi_i + \pi_j)\pi_k - \mu_3(\pi_i + \pi_j). \]

Thus

\[ \mu_i(\pi_i + \pi_j + \pi_k) > \mu_i\pi_k \geq 0 \]

because \( \mu_i \geq 0 \). We obtain \( \mu_i > 0 \), therefore \( x_j = x_k \).

Let us suppose now that 9 is verified. According to the equations of system 7 we have

\[ \frac{q_i}{q_j + q_k} = \frac{\lambda\pi_i - \mu_i + \mu_2}{\lambda(\pi_j + \pi_k) - \mu_2} > \frac{\pi_i}{\pi_j + \pi_k} \]

or

\[ \mu_2(\pi_i + \pi_j + \pi_k) > \mu_i(\pi_j + \pi_k) \geq 0. \]

Thus \( \mu_2 > 0 \) and hence \( x_i = x_j \).

Consequently, if the conditions 8 and 9 are verified, the riskless asset is the solution of 7.

We can also prove that the equations 8 and 9 form the necessary condition in order to a riskless asset would be the solution of system 7. At first we observe that the weights in the conditions 8 and 9 are being expressed in terms of 3 - dimensions portfolio problem. It would be useful for us to explain them in terms of 2 - dimensions than \( x_i = x_j \) or \( x_j = x_k \). In 3 – dimensions problem the decision weights have been defined as follows :

\[ q_i^3 = w(p_i), \quad q_j^3 = w(p_i + p_j) - w(p_i) \quad \text{and} \quad q_k^3 = 1 - w(p_i + p_j). \]

And for the dimension 2 we have

\[ q_{ij}^2 = w(p_i + p_j) \quad \text{and} \quad q_{jk}^2 = 1 - w(p_i + p_j) \quad \text{if} \quad x_i = x_j \]

\[ q_i^2 = w(p_i) \quad \text{and} \quad q_{jk}^2 = 1 - w(p_i) \quad \text{if} \quad x_j = x_k. \]

Thus, on the plan \( x_j = x_k \) we have \( q_i^3 + q_j^3 = w(p_i + p_j) = q_{ij}^2 \) and the weights associated to the best event are the same for both 2 and 3 dimensions : \( q_i^3 = q_{ij}^2 \). So, on the plan \( x_i = x_j \) the condition 8 becomes :

\[ \frac{q_{ij}}{q_k} > \frac{\pi_i + \pi_j}{\pi_k} \quad \text{(10)} \]
At the same time, on the plan \( x_j = x_k \) we have \( q_i^3 = q_i^2 \) and \( q_j^3 + q_k^3 = 1 - w(p_i) = q_{jk}^2 \). The equation 9 becomes:

\[
\frac{q_j}{q_{kj}} > \frac{\pi_j}{\pi_j + \pi_k}
\]

(11)

in order to prove that the condition 8 is a necessary condition we suppose that this condition is not satisfied:

\[
\frac{q_i + q_j}{q_k} \leq \frac{\pi_i + \pi_j}{\pi_k} \quad \text{or, we have} \quad \frac{q_j}{q_k} \leq \frac{\pi_j}{\pi_k}
\]

(12)

We can then indicate a risky portfolio which gives more satisfaction for the investor than a riskless asset. In fact, on the plan \( x_i = x_j \) hypothesis 12 means that the budget line is steeper than the indifference curves (Figure 9).

Figure 9

Thus, it is optimal for the person to invest all his money in \( e_k \). Let us consider portfolio \( P \) so that \( x_i = x_j = 0 \) and \( x_k = \frac{W_0}{\pi_k} \). We have \( E_h(P) = \frac{q_j w_0}{\pi_k} \) where \( h \) means that an agent deforms the real probabilities. We can prove that \( E_h(P) \geq \frac{W_0}{\pi_i + \pi_j + \pi_k} \) where the left term is the utility associated to the riskless asset. According to 12 we have the following inequality

\[
q_i \pi_k - (\pi_i + \pi_j)q_k \leq 0
\]

or

\[
(1 - q_k)\pi_k - (\pi_i + \pi_j)q_k \leq 0
\]

because \( q_i + q_k = 1 \). Thus
\[ \pi_k \leq (\pi_i + \pi_j + \pi_k) q_k \]

\[ \frac{q_k W_0}{\pi_k} \geq \frac{W_0}{(\pi_i + \pi_j + \pi_k)}. \]

This means that if inequality 8 is not satisfied we can point to a risky portfolio which gives more satisfaction for a person than a riskless asset. In other words the condition 8 is a necessary condition so that a riskless asset would be an optimal portfolio.

Using the same approach we prove that the condition 9 is also a necessary condition so that the investment in riskless asset would be optimal\(^9\). 

In conclusion, we proved the following theorem:

**Theorem 1**

Consider an agent who looks for resolving the following problem of maximization:

\[ \text{Max } x_i q_i + x_j q_j + x_k q_k \]

s.c \((x_i - x_{0i}) \pi_i + (x_j - x_{0j}) \pi_j + (x_k - x_{0k}) \pi_k = 0\)

\[ x_i \geq 0; \quad x_i \leq x_j; \quad x_j \leq x_k. \]

Where the weights were defined by

\[ q_i = w(p_i), \]

\[ q_j = w(p_i + p_j) - w(p_i) \text{ and} \]

\[ q_k = 1 - w(p_i + p_j). \]

A riskless asset is only one optimal portfolio in the set \( \bar{S}_{jk} \)\(^{10}\) if and only if

\[ \frac{q_i + q_j}{q_k} > \frac{\pi_i + \pi_j}{\pi_k} \]

and \( \frac{q_i}{q_j + q_k} > \frac{\pi_i}{\pi_j + \pi_k} \).

\(^9\) In this case the risky portfolio \( P \) has a form \( x_i = 0 \text{ et } x_j = x_k \) and \( E_k(P) = \frac{q_k W_0}{\pi_j + \pi_k} \geq \frac{W_0}{(\pi_i + \pi_j + \pi_k)} \)

\(^{10}\) \( \bar{S}_{jk} = \{W / 0 \leq x_i \leq x_j \leq x_k; \quad i, j, l = 1, 2, 3 \text{ et } i \neq j \neq k\} \)
This theorem proposes a sufficient and necessary condition in the set $S_{ijk}$.

Obviously, if a person chooses a riskless asset in each of six subsets $\tilde{S}_{ijk}$, this asset will be preferred by the agent for any other portfolio. Nevertheless, we prove that the condition of the theorem 1 cannot be verified in each subset $S_{ijk}$ simultaneously. To prove that, we rewrite the condition of the theorem 1 in terms of real probabilities $p_i, p_j, p_k$:

$$\frac{w(p_i + p_j)}{1-w(p_i + p_j)} > \frac{w_i + w_j}{w_k} \quad \text{and} \quad \frac{w(p_i)}{1-w(p_i)} > \frac{w_i}{w_j + w_k}$$

(13)

For instance, in the set $S_{123}$ these conditions take the following form

$$\frac{w(p_1 + p_2)}{1-w(p_1 + p_2)} > \frac{w_1 + w_2}{w_3} \quad \text{and} \quad \frac{w(p_1)}{1-w(p_1)} > \frac{w_1}{w_2 + w_3}.$$  

Finally, according to the theorem 1, a riskless asset would be an optimal portfolio in each $\tilde{S}_{ijk}$ if and only if the following system have a solution:

$$\begin{cases} 
\frac{w(p_1)}{1-w(p_1)} > \frac{\pi_1}{\pi_2 + \pi_3} \\
\frac{w(p_2)}{1-w(p_2)} > \frac{\pi_2}{\pi_1 + \pi_3} \\
\frac{w(p_3)}{1-w(p_3)} > \frac{\pi_3}{\pi_1 + \pi_2} \\
\frac{w(p_1 + p_2)}{1-w(p_1 + p_2)} > \frac{\pi_1 + \pi_2}{\pi_3} \\
\frac{w(p_1 + p_3)}{1-w(p_1 + p_3)} > \frac{\pi_1 + \pi_3}{\pi_2} \\
\frac{w(p_2 + p_3)}{1-w(p_2 + p_3)} > \frac{\pi_2 + \pi_3}{\pi_1} \\
\frac{w(p_1 + p_2 + p_3)}{1-w(p_1 + p_2 + p_3)} > \frac{\pi_1 + \pi_2 + \pi_3}{\pi}.
\end{cases}$$

(14)

We note $\pi = \pi_1 + \pi_2 + \pi_3$. After simple calculations the system 14 becomes
According to the three first equations we have \( w(p_1) + w(p_2) + w(p_3) > 1 \). But, each weighting function \( w \) must satisfy \( w(p_1) + w(p_2) + w(p_3) = 1 \). Therefore we proved that the necessary and sufficient condition of the theorem 1 cannot be verified in each subset \( S_{ijk} \) simultaneously. This means that at least one subset, that we shall call \( \tilde{S}_{ijk} \), exists on which the investment in a riskless asset in not optimal. Thus, in this particular subset \( \tilde{S}_{ijk} \) the agent will choose a risky portfolio \( P \) which gives him more satisfaction than a riskless asset. Because a riskless asset gives the same satisfaction on each subset \( S_{ijk} \), the risky portfolio \( P \) will be chosen by the agent at optimum. In conclusion, for a BPT - agent who maximizes his expected wealth in respect of weighting function, it would never be optimal to invest in a riskless asset.

4 Concluding Observations

The probability-weighting approach has been more and more frequently integrated in the models of decision making under risky alternatives. In this article we study the impact of this psychological phenomenon on the portfolio selection. Our reference point was the Behavioral Portfolio Theory developed by Shefrin and Statman (2000) which consider an investor who maximizes the expected wealth with deformed probabilities subject to a security constraint. We choose this theory as a reference point because the
optimal portfolio obtained by a BPT investor differs from the perfectly diversified optimal portfolio offered by mean-variance approach of Markowitz (1952). In fact, a BPT agent is able to invest some parts of his wealth in to lottery ticket. This risk seeking behavior, nevertheless, is not due to the probability weighting, contrary to what one can think. We prove, using an analytical and graphical approach, that the form of the optimal portfolio of an investor who transforms the objective probabilities is similar to the portfolio of an individual who does not do it. This does not mean that both will choose the same portfolio. We suggest that both will choose a risky portfolio belonging to the boundary of the set of all accessible assets. Therefore, there are non significant differences between the behaviors of two investors. Obviously, this result is not surprising in the case of a VNM agent who does not transform the real probabilities because he maximizes a linear function. Nevertheless, it is not the case for an investor who applies decision weights: his objective function is linear in some parts but not in the entire set of portfolios.

In conclusion, our main results can be put in two essential issues. First, we prove that the expected wealth criteria, even if the objective probabilities were deformed, is not enough for obtaining significantly different forms of portfolio. Not only probabilities but also future outcomes must be transformed. Secondly, it will not be difficult to prove that the particular form of a BPT investor is a result of maximization of a linear function on the particular set of portfolios. This is one of the many paths for our future research.
Annex 1

Case 1.1. The budget line is steeper than the indifference curves in the set $S_1$ (and hence in the $S_2$): $\frac{\pi_1}{\pi_2} > \frac{w(p_1)}{1-w(p_1)} > \frac{1-w(p_2)}{w(p_2)}$. In this case, the portfolio $(0, x_2)$ is the optimal solution in the set $S_1$. At the same time, the optimal portfolio of the set $S_2$ is positioned on the bisecting line. Finally, the optimal portfolio (in the $S_2$) takes form $(0, x_2)$. In other words, the agent invests only in one asset $e_2$.

Case 1.2. In both sets $S_1$ and $S_2$ the indifference curves are steeper than the budget line: $\frac{w(p_1)}{1-w(p_1)} > \frac{1-w(p_2)}{w(p_2)} > \frac{\pi_1}{\pi_2}$. The optimal solution of the set $S_1$ is positioned on the bisecting line and the one of $S_2$ takes the form $(x_1, 0)$. Finally, the optimal portfolio is $(x_1, 0)$. The agent invests all his wealth in the asset $e_1$.

Case 1.3. In the set $S_1$ the indifference curves are more steep than the budget line and the last is steeper than the indifference curves in the set $S_2$: $\frac{w(p_1)}{1-w(p_1)} > \frac{\pi_1}{\pi_2} > \frac{1-w(p_2)}{w(p_2)}$. In this case, a point on the bisecting line is the optimal solution in both sets $S_1$ and $S_2$. Thus, the optimal portfolio in the set $S_2$ is a riskless portfolio.
Annex 2

The following figure shows the case where the indifference curves of the set $S_2$ are more steep than the indifference curves of $S_1$: \[
\frac{w(p_1)}{1-w(p_1)} < \frac{1-w(p_2)}{w(p_2)}.
\]

![Figure 1](image.png)

We prove that the optimal agent’s choice in this case consists to invest all his wealth in only one asset as in the case of a VNM – investor. We consider three possibilities:

**Case 2.1.** The budget line is steeper than the indifference curves in the set $S_2$ (and hence in the set $S_1$): \[
\frac{w(p_1)}{1-w(p_1)} < \frac{1-w(p_2)}{w(p_2)} < \frac{\pi_1}{\pi_2}.\] In $S_1$ the optimal portfolio is $(0,x_2)$. In the set $S_2$ the optimal solution is positioned on the bisecting line. Finally, $(0,x_2)$ is the optimal portfolio in $\square_2^+$. (Figure 2).

**Case 2.2.** In both sets $S_1$ and $S_2$ the indifference curves are steeper than the budget line: \[
\frac{\pi_1}{\pi_2} < \frac{w(p_1)}{1-w(p_1)} < \frac{1-w(p_2)}{w(p_2)}\] Thus, the optimal solution of the set $S_1$ is positioned on the bisecting line and the one of $S_2$ takes the form $(x_1,0)$. Finally, $(x_1,0)$ is the optimal choice.
Case 2.1. \[ \frac{w(p_1)}{1-w(p_1)} < \frac{1-w(p_2)}{w(p_2)} < \frac{\pi_1}{\pi_2} \]

Case 2.2. \[ \frac{\pi_1}{\pi_2} < \frac{w(p_1)}{1-w(p_1)} < \frac{1-w(p_2)}{w(p_2)} \]

Case 2.3. \[ \frac{w(p_1)}{1-w(p_1)} < \frac{\pi_1}{\pi_2} < \frac{1-w(p_2)}{w(p_2)} \]

Figure 2

Case 2.3. In the set \( S_1 \) the budget line is steeper than the indifference curves and in the set \( S_2 \) the indifference curves are steeper than the budget line : \[ \frac{w(p_1)}{1-w(p_1)} < \frac{\pi_1}{\pi_2} < \frac{1-w(p_2)}{w(p_2)} \]. In this case, \((0, x_2)\) is the optimal solution in the set \( S_1 \) and \((x_1, 0)\) is the optimal portfolio in \( S_2 \). Therefore, the optimal portfolio is always a risky portfolio (on the figure 2 the portfolio \((x_1, 0)\) gives more satisfaction to agent than \((0, x_2)\)).


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